Partial fractions and Integration of Rational function:-

Defn:- Any function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two different polynomials are called as 'Rational function'.

Generally, polynomials can be expressed like

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \; a_n \neq 0$$

is said to be a polynomial of degree 'n'.

example:-

1. $P(x) = x^3 + 3x^2$ — degree = 3 → cubic polynomial
2. $P(x) = 2 = 2 \cdot x^0$ — degree = 0 → constant
3. $P(x) = x + 5$ — degree = 1 → Linear
4. $P(x) = x^2 + 2x + 5$ — degree = 2 → Quadratic

Rational function <
- Proper Rational, if degree $P(x)$ is less than deg $Q(x)$
- Improper Rational, if deg $P(x) >$ deg $Q(x)$

PROCEDURE TO BREAK A FRACTION INTO RATIONAL FRACTION:-

Step-1:- Consider a proper rational function ($\deg P(x) < \deg Q(x)$), then they can be expressed into simpler forms. So four different cases arises.

Case-I Factorising the denominator into real factors, we get that non-repeated distinct linear factor arises. Then for each non-repeated linear factor, the corresponding partial fraction will be $\frac{A}{ax+b}$, if A is constant.

ex- $f(x) = \frac{2x}{x^2+5x+6}$

On factorising $Q(x) = x^2+5x+6$

we get that $x^2+5x+6 = (x+3)(x+2)$

Non repeated, linear

∴ So the partial fraction will be

$$\frac{2x}{x^2+5x+6} = \frac{A_1}{x+3} + \frac{A_2}{x+2}$$

A_1, A_2 are constant, we have to evaluate

Check

$\deg P(x) = 1$

$\deg Q(x) = 2$

$P(x) = 2x$

$Q(x) = x^2+5x+6$

∴ $\deg P(x) = 1 <$

$\deg (Q(x)) = 2$

⇒ Proper fraction

case-II For any linear factor appearing 'r' times in the factorisation of $Q(x)$, the corresponding partial fractions will be sum $\frac{A_r}{(ax+b)^r}$, $(r=1,2,3,\dots)$

example $g(x) = \frac{2x+5}{(x+2)^3 (x+5)}$

3 times Repeated, Distinct

So the partial fraction will be

$$\frac{2x+5}{(x+2)^3} = \frac{A_1}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{(x+2)^3} + \frac{A_4}{(x+5)}$$

Case-III For any non-repeated quadratic factor (cx^2+ax+b) in the denominator, the corresponding partial fraction will be of the form $\frac{Ax+B}{cx^2+ax+b}$

ex:- $h(x) = \frac{2x+7}{(x+1)(x^2+x+2)}$

$$= \frac{A_1}{x+1} + \frac{Bx+C}{x^2+x+2}$$

Case-IV To a repeated quadratic factor cx^2+ax+b 'r' times, the corresponding pf are $\frac{A_r x+B_r}{(cx^2+ax+b)^r}$, $(r=1,2,\dots)$

ex:- $$\frac{2x+3}{(x^2+2x+5)^2} = \frac{a_1x+b_1}{x^2+2x+5} + \frac{a_2x+b_2}{(x^2+2x+5)^2}$$

Step-II

Now evaluate the both side by making an identity of RHS and LHS.

Step-III

Comparing both side of equation and co-efficients of variable, we will get equations, which will gives us the value of required constants.

Step-IV

Substitute the various constant at their PFs, to get solution. Then proceed for Integration.

Examples Based On the Partial Fraction Method: -

1) Integrate $\int \frac{x^2 - x - 1}{(x+1)(x-1)(x-2)} dx$.

Ans:- clearly, it is a proper fraction with three distinct linear factors $x+1$, $x-1$, $x-2$. So the partial fraction will be

$$\frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{x-2} = \frac{x^2 - x - 1}{(x+1)(x-1)(x-2)} \quad \text{— (1)}$$

Taking LCM in LHS, we will get that

$$\therefore \frac{x^2 - x - 2}{(x+1)(x-1)(x-2)} = \frac{A_1(x-1)(x-2) + A_2(x+1)(x-2) + A_3(x+1)(x-1)}{(x-1)(x-2)(x+1)}$$

Canceling denominator of both side, we get

$$x^2 - x - 2 = A_1(x-1)(x-2) + A_2(x+1)(x-2) + A_3(x+1)(x-1)$$

$$= A_1(x^2 - 3x + 2) + A_2(x^2 - x - 2) + A_3(x^2 - 1)$$

$$= x^2(A_1 + A_2 + A_3) + x(-3A_1 - A_2) + (2A_1 - 2A_2 - A_3)$$

[$\therefore$ we've collected co-efficients of x^2, x, constants from each factor.]

Now comparing the both sides,

$$x^2 = x^2(A_1 + A_2 + A_3) \Rightarrow A_1 + A_2 + A_3 = 1 \quad \text{— (2)}$$

$$-x = x(-3A_1 - A_2) \Rightarrow -3A_1 - A_2 = -1 \quad \text{— (3)}$$

$$-2 = 2A_1 - 2A_2 - A_3 \Rightarrow 2A_1 - 2A_2 - A_3 = -2 \quad \text{— (4)}$$

$\therefore$ solving (2), (3), (4) by elimination process,

we get that $A_1 = \frac{1}{6}$

$A_2 = \frac{1}{2}$

$A_3 = \frac{1}{3}$

$\therefore$ Substituting the values of A_1, A_2, A_3 in eqn (1)

we get that

$$\therefore \int \frac{x^2 - x - 1}{(x+1)(x-1)(x-2)}\,dx = \int \frac{dx}{6(x+1)} + \int \frac{dx}{2(x-1)} + \int \frac{dx}{3(x-2)}$$

$$= \frac{1}{6}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{3}\ln|x-2| + c$$

D.

Exercises Integrate the following

1. $\int \frac{dx}{x^2 - a^2}$
2. $\int \frac{dx}{a^2 - x^2}$
3. $\int \frac{4x-9}{x^2-5x+6}\,dx$
4. $\int \frac{3x}{(x-4)(x+2)}\,dx$
5. $\int \frac{5x-12}{(x+3)(x-1)}\,dx$.
6. $\int \frac{4x+5}{x^2+x-2}\,dx$.

— x —

Example - 2

Integrate $\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx$

Ans:- Clearly in denominator there is a quadratic term (x^2+2) & linear factor $(x-1)$ appears. So the partial fraction will be

$$\frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1} \quad \text{—— (1)}$$

Taking LCM of RHS and equating both sides, we get

$$= \frac{(Ax+B)(x-1) + C(x^2+2)}{(x^2+2)(x-1)}$$

Cancelling denominator of both sides, we get that

$$\Rightarrow \frac{2x^2+x+3}{(x^2+2)(x-1)} = \frac{Ax^2+Bx-Ax-B+Cx^2+2C}{(x^2+2)(x-1)}$$

$$\Rightarrow 2x^2+x+3 = (A+C)x^2 + (B-A)x + (2C-B)$$

Comparing the terms involving x^2, x, const. from both side

$$2x^2 = (A+C)x^2 \Rightarrow 2 = A+C \quad \text{—— (2)}$$

$$x = (B-A)x \Rightarrow 1 = B-A \quad \text{—— (3)}$$

$$3 = 2C-B \Rightarrow 3 = 2C-B \quad \text{—— (4)}$$

eqn (2) + eqn (3) $\Rightarrow$

$\cancel{A} + C + B - \cancel{A} = 2 + 1$

$\Rightarrow C + B = 3$ — (5)

from (4) & (5)

$2C - \cancel{B} + C + \cancel{B} = 3 + 3$

$\Rightarrow 3C = 6 \Rightarrow \boxed{C = 6/3 = 2}$

from (2), $A + C = 2$

$\Rightarrow A + 2 = 2$

$\Rightarrow \boxed{A = 0}$

from (3), $B - A = 1$

$\Rightarrow B - 0 = 1$

$\Rightarrow \boxed{B = 1}$

$\therefore$ Putting the values of A, B, C in eqn (1), we get that

$$\int \frac{2x^2+x+3}{(x^2+2)(x-1)}\,dx = \int \frac{dx}{x^2+2} + \int \frac{2\,dx}{x-1}$$

$$= \int \frac{dx}{x^2+2} + 2\int \frac{dx}{x-1}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + 2 \ln|x-1| + c \quad \text{D.}$$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{dx}{x+a} = \ln|x+a| + c$

Exercise

(1) $\int \frac{2x^2+x-4}{(x^2+1)(x-2)}\,dx$

(2) $\int \frac{4x^2-x+3}{(x^2+1)(x-2)}\,dx$

(3) $\int \frac{x}{(x-1)(x^2+4)}\,dx$

(4) $\int \frac{2x+9}{(x+3)^2}\,dx$

(5) $\int \frac{dx}{x^2-5}$

(6) $\int \frac{e^x}{e^{2x}+3e^x+1}\,dx$

(7) $\int \frac{x^2}{(x^2+2)(x^2+3)}\,dx$

DEFINITE INTEGRATION

If $f(x)$ is any continuous function in $[a,b]$, then $\int_a^b f(x)\,dx$ is called the definite integral between the limit a and b.

$a \rightarrow$ lower / Inferior limit

$b \rightarrow$ upper / Superior limit

Fundamental Theorem Of Integral Calculus :-

If $f(x)$ is continuous function in $[a,b]$ and $F(x)$ is anti-derivative of $f(x)$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

example

1> $\int_0^1 x^2\,dx = \left|\frac{x^3}{3}\right|_0^1 = \left(\frac{x^3}{3}\right)_{x=1} - \left(\frac{x^3}{3}\right)_{x=0}$

$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3} - 0 = \frac{1}{3}$ Ans.

2> $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

Exercises

1. $\int_2^4 [x]\,dx$.
2. $\int_0^1 e^x\,dx$
3. $\int_0^1 e^{5x+3}\,dx$
4. $\int_{-1}^2 x^4\,dx$
5. $\int_0^1 x^{1/3}\,dx$
6. $\int_1^4 \frac{dx}{\sqrt{x}}$
7. $\int_0^1 (2x+1)^4\,dx$
8. $\int_0^{\pi/4} \cos 2x\,dx$
9. $\int_0^{\pi/2} (\cos x - \sin x)\,dx$
10. $\int_0^{\pi/2} \frac{dx}{1+\cos x}$
11. $\int_0^{\pi/4} \cos^2 x\,dx$
12. $\int_0^{\pi/4} \sin^5 x \cdot \cos x\,dx$
13. $\int_0^{\pi/4} \frac{e^z}{e^{2z}+4}\,dz$
14. $\int_{-4}^3 |x|\,dx$
15. $\int_{-1}^2 (2x-1)(x-2)\,dx$
16. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Elementary Properties Of Definite Integral :-

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

By altering the upper & lower limits, the definite Integral is multiplied by minus one.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$= -\{F(a) - F(b)\} \quad (\because \text{Taking minus outside})$$
$$= -\int_b^a f(x)\,dx.$$

2. $$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$$

Definite Integral is independant of the symbol of variable.

In each of the above case, the result will be

$F(b) - F(a)$.

3. If in the interval $[a, b]$, there is a point 'k' exist such that value at 'k' can be found, then

$$\int_a^b f(x)\,dx = \int_a^k f(x)\,dx + \int_k^b f(x)\,dx.$$

$$\Rightarrow F(b) - F(a) = \cancel{F(k)} - F(a) + F(b) - \cancel{F(k)}$$

$$\Rightarrow F(b) - F(a) = F(b) - F(a)$$

Example:- find the definite Integral of [By property 3]

$$\int_{-3}^4 |x|\,dx = \int_{-3}^0 |x|\,dx + \int_0^4 |x|\,dx$$

(breaking interval so that postive values in one integral & negatives of 'x' in another

we know $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ of 'x'

$$= \int_{-3}^0 -x\,dx + \int_0^4 x\,dx = \int_0^4 x\,dx + \int_0^{-3} x\,dx \quad \text{(By property 1)}$$

$$= \int_0^4 x\,dx + \int_0^{-3} x\,dx$$

$$= \left.\frac{x^2}{2}\right|_0^4 + \left.\frac{x^2}{2}\right|_0^{-3}$$

$$= \frac{4^2}{2} - \frac{0}{2} + \frac{(-3)^2}{2} - \frac{0}{2}$$

$$= \frac{16}{2} - 0 + \frac{9}{2} - 0$$

$$= \frac{16}{2} + \frac{9}{2} = \frac{25}{2}$$

Property-4

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Proof:- Consider $\int_0^a f(a-x)\,dx$.

By taking $a - x = z$, $dz = -dx$.

So for $x=0$, $z = a-0 = a$

for $x = a$, $z = a - a = 0$

$$\therefore I = \int_a^0 f(z)(-dz)$$

$$= \int_a^0 -f(z)\,dz$$

$$= \int_0^a f(z)\,dz$$

$$= \int_0^a f(x)\,dx$$

$\int_0^a f(x)\,dx = \int_a^0 -f(x)\,dx$

(By property -1)

(By property - 2)

Definite Integral is independant of symbol of variable)

$\therefore$ So we can use 'x' in place of z.

Property - 5

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

Proof:-

Any function is said to be even function if it satisfies

$$f(x) = f(-x)$$

let $y = x^2 = f(x)$

Then $f(-x)$ is the value by putting $-x$.

$$\therefore f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore$ so it is even

Some even functions are $\cos x$, x^{2n}.

Any function $f(x)$ is said to be an odd function, if it satisfies
$$f(-x) = -f(x)$$

Some odd functions are $\sin x$, $\tan x$, x^n where n is odd.

Let $f(x) = \tan x$

Put $-x$, $f(-x) = \tan(-x) = \dfrac{\sin(-x)}{\cos(-x)}$

$$= \frac{-\sin x}{\cos x}$$

$$= -\tan x$$

$$= -f(x)$$

$\Rightarrow$ Tan x is an odd function.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx. \quad \text{— (1)}$$

$$= I_1 + I_2$$

for I_1, Take $x = -u \Rightarrow dx = -du$

for $x = -a, \quad u = a$

$x = 0, \quad u = 0$

$$\Rightarrow I_1 = \int_{a}^{0} f(-u) - du = -\int_{a}^{0} f(-u)\,du$$

$$= \int_{0}^{a} f(-u)\,du$$

$$= \int_{0}^{a} f(-x)\,dx$$

So eqn (1) becomes,

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx.$$

If f is even, $f(-x) = f(x)$, so it reduces

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Now if f is odd, Then $f(-x) = -f(x)$, Then

$$= \int_{0}^{a} -f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx - \int_{0}^{a} f(x)\,dx$$

$$= 0$$

D.

ex-1 Integrate $\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx$

Ans:- As we know that $\sin x$ is odd function, so by property

$$\int_{-a}^{a} f(x)dx = 0 \quad \text{if } f(x) \text{ is odd}$$

$$= 2\int_0^a f(x)\,dx, \quad \text{if } f(x) \text{ is even.}$$

ex-2 $\int_{-\pi/4}^{\pi/4} \cos^3 x\, dx$

Ans:- As we know that $\cos x$ is even function, Then

$$\int_{-\pi/4}^{\pi/4} \cos^3 x\, dx = 2\int_0^{\pi/4} \cos^3 x\, dx.$$

$$= 2\int_0^{\pi/4} \cos^2 x \cdot \cos x\, dx.$$

$$= 2\int_0^{\pi/4} (1-\sin^2 x)\cos x\, dx.$$

Take $z = \sin x$, $dz = \cos x\, dx$.

for $x=0$, $z = 0$; $x = \pi/4$, $z = 1/\sqrt{2}$

$$= 2\int_0^{1/\sqrt{2}} (1-z^2)\,dz$$

$$= 2\left\{\int_0^{1/\sqrt{2}} dz - \int_0^{1/\sqrt{2}} z^2 dz\right\}$$

$$= 2\left\{\left| z \right|_0^{1/\sqrt{2}} - \left.\frac{z^3}{3}\right|_0^{1/\sqrt{2}}\right\}$$

$$= 2\left\{\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right\}$$

$$= \frac{2}{\sqrt{2}} - \frac{2}{6\sqrt{2}} = \frac{5\sqrt{2}}{6} \quad D.$$

ex-3 Integrate $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

Ans:- Let $I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta$

$$= \int_0^{\pi/4} \ln\left(1+\tan\left(\pi/4-\theta\right)\right)d\theta$$

using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \cancel{\log}\ \ln 2\, d\theta - \int_0^{\pi/4} \cancel{\log}\ \ln(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \cancel{\log 2}\ \ln\, d\theta - I$$

$$\therefore \quad I = \cancel{\log 2}\ \ln 2 \int_0^{\pi/4} d\theta - I$$

$$= \cancel{\log 2}\ \ln 2\, [\theta]_0^{\pi/4} - I$$

$$= \cancel{\frac{\pi \log 2}{4} - I}$$

$$= \frac{\pi \ln 2}{4} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4} \ln 2$$

$$\Rightarrow \quad \boxed{I = \frac{\pi}{8} \ln 2}$$

D

Example-3 Integrate $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

Ans:- Let $I = \int_0^{\pi/2} \frac{dx}{1+\tan x}$

$$= \int_0^{\pi/2} \frac{dx}{1 + \frac{\sin x}{\cos x}}$$

$$= \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x} \quad \text{—— (1)}$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right) + \sin\left(\frac{\pi}{2}-x\right)}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx. \quad \text{—— (2)}$$

Adding (1) & (2), we get that

$$2I = \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x} + \int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}\, dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = \int_0^{\pi/2} dx = x\Big]_0^{\pi/2} = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

D.

Exercise

Q1> $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$

Q2> $\int_0^{1} \frac{\ln(1+x)}{1+x^2}\, dx$

Q3> $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$.

Q4> $\int_0^{\pi/2} \frac{1}{1+\cot x}\, dx$.

Geometrical Significance Of Definite Integration:-

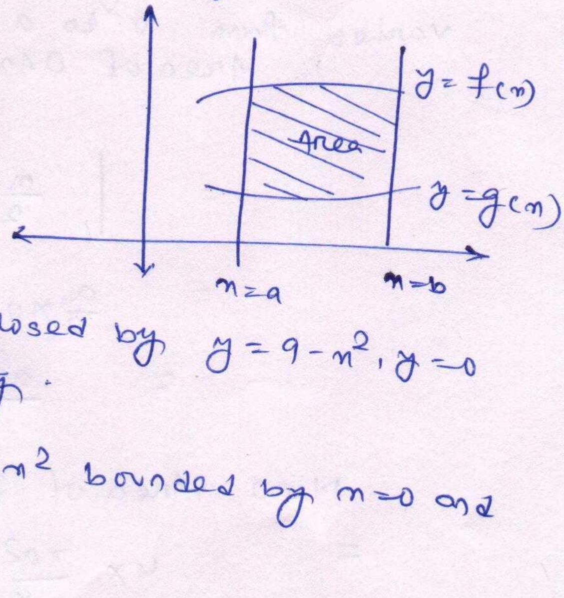
The definite Integral $\int_a^b f(x)\,dx$ represents the area under the curve $y=f(x)$, above the x-axis between the ordinate $x=a$ and $x=b$.

Area Between two curves:-

If there are two curves $y=f(x)$ and $y=g(x)$ where $g(x)<f(x)$ in $[a,b]$, then the area under the curve is given by

$$A=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$$

$$=\int_a^b [f(x)-g(x)]\,dx.$$



Example find the area of the region enclosed by $y=9-x^2$, $y=0$ and the ordinates $x=0$ and $x=2$ ~~is given by~~.

Ans:- The area under the curve $y=9-x^2$ bounded by $x=0$ and $x=2$ is given by

$$A=\int_0^2 (9-x^2)\,dx$$

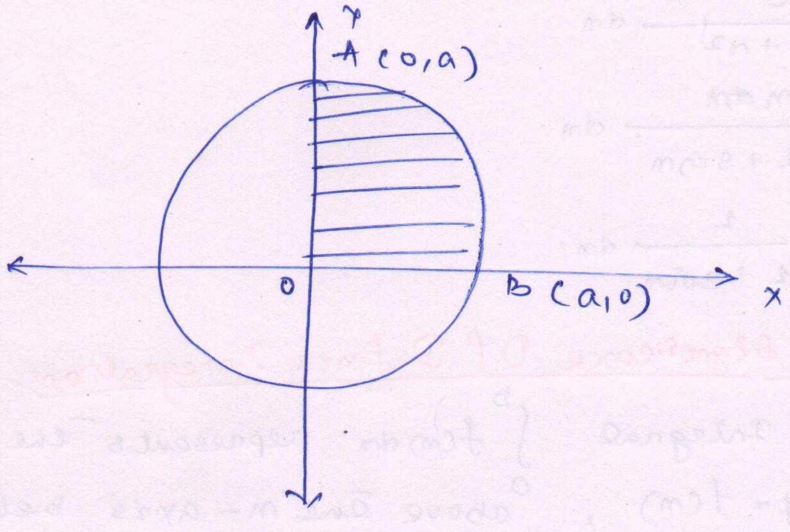
$$=\int_0^2 9\,dx-\int_0^2 x^2\,dx$$

$$=9x\Big|_0^2-\frac{x^3}{3}\Big|_0^2$$

$$=18-\frac{8}{3}=\frac{46}{3} \text{ sq unit}$$

2> find the area of the circle $x^2+y^2=a^2$

Ans:- As we know that circle is symmetrical about y-axis. i.e. & each semi circle have an area twice of its quadrant portion.



Total area of circle $= 4\times$ Area of region OAB.

Now $x^2+y^2=a^2$

$\Rightarrow y^2 = a^2-x^2$

$\Rightarrow y = \sqrt{a^2-x^2}$ in the 1st quadrant. Also 'x' varies from 0 to a along x-axis.

$\therefore$ Area of OAB $= \int_0^a \sqrt{a^2-x^2}\,dx$

$= \left| \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right|_0^a$

$= \frac{a}{2}\times 0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - 0$

$= \frac{a^2\cdot\pi}{4} = \frac{\pi a^2}{4}$ sq unit.

Now Area of the circle $= 4\times$ Area of OAB region.

$= 4\times\frac{\pi a^2}{4} = \pi a^2$ sq unit.

Exercise

Q1> Determine the area bounded by the curve $y^2=x$, $x=0$, $y=1$.

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Q2> find the area bounded by the curve $xy=c^2$, the x-axis, $x=2$, $x=3$

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Q3> find the area bounded by the curve $y=\cos x$, $y=\sin x$ and $x=0$?