A LECTURE NOTE
ON
TH.3 – ENGINEERING MATHEMATICS-I
SEMESTER -1

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DIFFERENTIAL EQUATIONS

Defn:- An equation that involves differentials or derivatives is known as "Differential Equation".

Some example of differential equation

① $x\,dx + y\,dy = 0$

② $\dfrac{dy}{dx} + 3y = 5$

③ $\dfrac{dy}{dx} + 5y = e^x$

$\left.\right\}$ (ODE)

④ $z\dfrac{\partial y}{\partial z} + x\dfrac{\partial y}{\partial x} = \sin y$ (PDE)

⑤ $\left(\dfrac{dy}{dx}\right)^2 + x^2 = \sqrt{y}$ (ODE)

⑥ $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = e^x$ (ODE)

⑦ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (PDE)

Differential Equation

↓ Ordinary Differential Equation (ODE)
↓ Derivatives of a single variable is used

↓ Partial Differential Equation (PDE)
↓ Partial Derivatives of more than one variable is used.

Order of an ODE :-

The order of the highest ordered derivatives occuring in it is called as order of given ODE.

In example 1, 2, 3, 5, 4, the derivatives used are first order derivatives.

$\Rightarrow$ Order of each differential equation is 1.

In example 6, the derivative used is $\dfrac{d^3}{dx^3}$ is of 3rd order

$\Rightarrow$ $\boxed{\text{Order} = 3}$

In example 7, partial derivative of second order are used,

$\Rightarrow$ Order = 2.

Degree of ODE:-

The degree of the differential equation is the highest integral power of the derivative that determines the order of the equation after removing all fractional power, indices.

ex-1 $\quad x\,dx + y\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \frac{dy}{dx} - \frac{x}{y} = 0 \Rightarrow \left(\frac{dy}{dx}\right)^{1} - \frac{x}{y} = 0$$

Term decides order

Here the above encircle term have integral power = 1

$\Rightarrow$ By def^n degree = 1

So ex-1, 2, 3, 6, 7 have degree = 1. Try to find order, degree in each example by yourself

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = e^x$$

- No term contains fractional power.
- Here order = 3, so degree will be the power of this term $\left(\frac{d^3y}{dx^3}\right)^{1} = \frac{d^3y}{dx^3}$

$\therefore$ Degree = 1.

ex-5

$$\left(\frac{dy}{dx}\right)^2 + x^2 = \sqrt{y} \leftarrow \text{Radical}$$

- In RHS, there is a term $\sqrt{y}$, contains fractional power

$$\sqrt{y} = y^{1/2}$$

- So we will try make it integer power.

$$\left(\left(\frac{dy}{dx}\right)^2\right)^2 + (x^2)^2 = (\sqrt{y})^2$$ [Squaring will give an integer power

$$\Rightarrow \left(\frac{dy}{dx}\right)^4 + x^4 = y$$

$\leftarrow$ Term decide the order = Highest order derivative

$\leftarrow$ Degree will be its power

$\therefore$ Order = 1, Degree = 4

SOLUTION OF A DIFFERENTIAL EQUATION:-

A relation like $y = f(x)$ or $f(x,y) = 0$ between the variables is called solution of a given differential equation, if it reduces the equation to an identity when substituted into it.

Suppose $y = e^x$ is given to be a solution of the ODE

$$\frac{dy}{dx} = y \quad \text{--- (1)}$$

Now taking $y = e^x$

$$\frac{dy}{dx} = e^x$$

So eq^n (1) becomes,

$$e^x = e^x \quad \leftarrow \text{Identity.}$$

The same result also can be obtained by taking

$y = 5e^x$, $y = 7e^x$, $y = e^x$, // check yourself

$\Rightarrow$ In general, $y = ce^x$ (c = const) can satisfy the eq^n (1) known as "General solution".

$\Rightarrow$ $y = 5e^x, 7e^x, e^x$ are obtained by putting different value of c is known as Particular solution.

Variable Separation Method:-

Suppose $\frac{dy}{dx} = f(x)\,g(y)$ be any ODE. Then the solutions can be obtained by separating variables in two sides i.e. function of x with dx, function of y with dy.

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Now integrating both sides, we will get

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + c$$

is the required general solution.

example - 1

solve $\frac{dy}{dx} = x^2 + 2x + 5$

Ans:- The above equation can be expressed as

$$\underbrace{dy}_{\text{Terms contains only 'y'}} = \underbrace{(x^2 + 2x + 5)\,dx}_{\text{Term contains only 'x'}}$$

Integrating both sides, we get that

$$\int dy = \int (x^2 + 2x + 5)\,dx$$

$$\Rightarrow y = x^3/3 + x^2 + 5x + c$$

is the required general solution.

example solve

$$\frac{dx}{dy} = \frac{x+1}{y+1}$$

Ans:- The above equation can be written as

$$\frac{dx}{x+1} = \frac{dy}{y+1}$$

Integrating both sides, we get that

$$\int \frac{dx}{x+1} = \int \frac{dy}{y+1}$$

$$\Rightarrow \ln(x+1) = \ln(y+1) + \ln c$$

$$\Rightarrow \ln\left(\frac{x+1}{y+1}\right) = \ln c$$

$$\Rightarrow \boxed{\frac{x+1}{y+1} = c}$$ is the general solution.

Refer exercise - 14 (a), pg - 187 for problems relating the above topic on ODE.

All question's of 1, 3, 4, 5, 7

LINEAR DIFFERENTIAL EQUATION:-

A differential equation is said to be linear if it satisfi

(i) Differential co-efficient & dependant variable are not multipl together.

(ii) degree is equals to 1.

The general form of linear ODE is given by

$$\frac{dy}{dx} + Py = Q$$

where P, Q are functions in 'x'.

Procedure to find general solution of Linear Differential Eqn:-

Step-I Write the equation in standard form

$$\frac{dy}{dx} + Py = Q$$

Step-II Determine Integrating factor (I.f) by

$$I.f = e^{\int P dx}$$

Step-III Then perform $y.(If) = \int Q.If\, dx + c$ to obtain the general solution.

— x —

Example Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$. (2019 W) (5 Marks)

Ans:- The above equation can be expressed as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

Here $P(x) = \frac{2x}{1+x^2}$ $\quad Q(x) = \frac{4x^2}{1+x^2}$

$$If = e^{\int P dx} = e^{\int \frac{2x dx}{1+x^2}} = e^{\ln(1+x^2)}$$

(Take $t = 1+x^2$, $dt = 2x dx$)

$$= e^{\ln(1+x^2)} = 1+x^2$$

$\therefore$ The solution of the linear equation is

$$y \times (1+x^2) = \int \frac{4x^2}{1+x^2} \times (1+x^2)dx + c$$

$$\Rightarrow y(1+x^2) = \int 4x^2 dx + c$$
$$= \frac{4x^3}{3} + c$$
$$\Rightarrow y = \frac{4}{3}\left(\frac{x^3}{1+x^2}\right) + c(1+x^2)^{-1}$$

is the required general solution.

Example - 2 Solve

$$4\frac{dy}{dx} + 8y = 5e^{-3x}$$

Ans:- The above equation can be written as

$$\frac{dy}{dx} + 2y = \frac{5}{4}e^{-3x}.$$

Here $P = 2$, $Q = \frac{5}{4}e^{-3x}$

$$I.F = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$$

The solution of linear equation is given by.

$$y \cdot e^{2x} = \frac{5}{4}\int e^{-3x} \cdot e^{2x} dx + c$$
$$= \frac{5}{4}\int e^{-x} dx + c$$
$$= -\frac{5}{4}e^{-x} + c$$
$$\Rightarrow \boxed{y = -\frac{5}{4}e^{-3x} + ce^{-2x}}$$

Previous Year Question:-

Q1/ Solve $\frac{dy}{dx} + y\tan x = \sec x$ (10 Mark)

Q2/ $\frac{dy}{dz} = \frac{\sqrt{1-y^2}}{\sqrt{1-z^2}}$ (5 Mark)

Q3/ Solve the differential eqn $x(1+y^2)dx + y(1+x^2)dy = 0$

Q4/ Solve $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{x^3}{1+x^2}$ (7 Mark)

Q5/ Solve: $\frac{dy}{dx} + (\sec x)y = \tan x$ (5 Mark)

Q6/ Solve: $e^x \tan y\, dx + (1+e^x)\sec^2 y\, dy = 0$

Q7/ Solve: $(1+x^2)\frac{dy}{dx} + 2xy = x^3$

—x—